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SPATIAL LOCAL SOLUTIONS OF THE NAVIER–STOKES EQUATIONS

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This paper considers solutions of the Navier–Stokes equations polynomial in the coordinates, which are called local solutions. For an incompressible fluid, all higher-order terms (sums of higher-order monomials) of degree 2 are found and it is proved that nontrivial axisymmetric higher-order terms of degree higher than 2 do not exist. Nonsolenoidal axisymmetric solutions are listed, which can be treated as steady-state barotropic gas flows in a potential external-force field. All elliptic vortices generalizing the well-known Kirchhoff solution are calculated. All solutions of degree 3 with the higher-order term of partial form are found. Some of these solutions break down in a finite time regardless of the value and sign of viscosity.

Key words: *viscous fluid, polynomial, local solution, higher-order term, elliptic vortex.*

The solutions which are polynomials in spatial variables well be referred to as local solutions. Locally, i.e., in a small vicinity of a point, a smooth function is defined approximately by the Taylor formula. The solutions of the Navier–Stokes equations are analytical in the coordinates of a point in space [1] and are therefore expanded in a Taylor series in the vicinity of each interior point in the domain of definition. These series, however have no application since their radius of convergence is very small. For the plane stationary case, Nowak [2] estimated the radius of convergence $\mathrm{const}\cdot\nu$ ($\nu = \mathrm{Re}^{-1}$ is the dimensionless fluid viscosity; Re is the Reynolds number). For $\nu = 0$, there are examples of nonanalytical solutions [3]; therefore, this estimate can hardly be improved. Generally, a Taylor series segment approximates a solution only in the indicated negligibly small vicinity. Local solutions are free of this drawback since they are exact solutions over the entire space and defined for all Reynolds numbers. All plane local solutions are found in [3]. The present paper is devoted to finding solutions describing spatial flows.

1. Solutions Which are Polynomial in $\boldsymbol{x}$. The velocity $\boldsymbol{u} = (u_1, u_2, u_3)$ and the pressure p of an incompressible viscous fluid of density 1 as functions of a point in space $\boldsymbol{x} = (x_1, x_2, x_3)$ and time t satisfy the Navier–Stokes equations

$$\nabla\cdot\boldsymbol{u} = 0, \qquad \frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \nu\Delta\boldsymbol{u}, \tag{1.1}$$

where ν is the dimensionless fluid viscosity. If the velocity is a polynomial in x_1, x_2, x_3, the pressure is also a polynomial (since ∇p is a polynomial). Therefore, we eliminate the pressure from the equation of motion applying the rot operation. The obtained Helmholtz equation is written in a coordinate system rotating around its origin:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} - \nu\Delta\boldsymbol{\omega} + 2(\boldsymbol{o}\cdot\nabla)\boldsymbol{u} - 2\dot{\boldsymbol{o}} = 0. \tag{1.2}$$

Here $\boldsymbol{\omega} = \mathrm{rot}\,\boldsymbol{u} = \nabla\times\boldsymbol{u}$ is the vortex; $\boldsymbol{o}$ is the angular velocity of rotation of the coordinate system; the dot above the letter denotes the time derivative.

The solution is sought in the form

$$\boldsymbol{u} = \sum_{k=1}^{n}\boldsymbol{u}_k \qquad (n \geq 1), \tag{1.3}$$

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where $\boldsymbol{u}_k$ is a homogeneous polynomial in $\boldsymbol{x}$ of degree k with coefficients arbitrarily dependent on t. It should be noted that $\boldsymbol{u}_0 = 0$, i.e., $\boldsymbol{u} = 0$ at $\boldsymbol{x} = 0$. This condition can be satisfied by transforming to a translationally moving coordinate system whose origin is on a fluid particle. The sum of higher-orders monomials $\boldsymbol{u}_n$ will be called the higher-order term. It is convenient to perform expansion (1.3) in a rotating coordinate system in which the higher-order term $\boldsymbol{u}_n$ has the simplest form. In the transformation to the rotating coordinate system, the uniformity property and the degree of the polynomials $\boldsymbol{u}_k$ are retained (the translational velocity $\boldsymbol{u}_1$ is subtracted from $\boldsymbol{o} \times \boldsymbol{x}$). Substituting the sum (1.3) into the fluid incompressibility condition (1.1) and Eq. (1.2) and equating each of the homogeneous term of different degrees to zero, we obtain

$$\nabla \cdot \boldsymbol{u}_k = 0 \qquad (k = 1, \ldots, n); \tag{1.4}$$

$$\frac{\partial \boldsymbol{\omega}_k}{\partial t} + \sum_{i+j=k+1} ((\boldsymbol{u}_i \cdot \nabla)\boldsymbol{\omega}_j - (\boldsymbol{\omega}_j \cdot \nabla)\boldsymbol{u}_i) - \nu \Delta \boldsymbol{\omega}_{k+2} + 2(\boldsymbol{o} \cdot \nabla)\boldsymbol{u}_k - 2\dot{\boldsymbol{o}}\delta_{1k} = 0$$
$$(1 \le i, j \le n; \quad k = 1, \ldots, 2n-1), \tag{1.5}$$

where $\boldsymbol{\omega}_k = \nabla \times \boldsymbol{u}_k$, $\boldsymbol{u}_k = \boldsymbol{\omega}_k = 0$ for $k < 1$ or $k > n$; δ_{ij} is the Kronecker symbol ($\delta_{ii} = 1$, $\delta_{ij} = 0$ for $i \neq j$).

2. Higher-Order Term. The case $n = 1$ is considered in Sec. 4. In this section, we set $n \ge 2$. Then, Eqs. (1.4) for $k = n$ and (1.5) for $k = 2n - 1$ coincide with the equations of steady-state motion for an inviscid incompressible fluid:

$$\nabla \cdot \boldsymbol{u}_n = 0; \tag{2.1}$$

$$(\boldsymbol{u}_n \cdot \nabla)\boldsymbol{\omega}_n - (\boldsymbol{\omega}_n \cdot \nabla)\boldsymbol{u}_n = 0. \tag{2.2}$$

In (2.1) and (2.2), the time t is included only as a parameter; therefore, in this section, it is not indicated in the arguments. Thus, it is required to find homogeneous polynomials in $\boldsymbol{x}$ that satisfy Eqs. (2.1) and (2.2). The following obvious solutions (or solutions having the same form in an appropriate coordinate system) will be called trivial:

(a) potential flow $\boldsymbol{u}_n = \nabla\varphi$, $\Delta\varphi = 0$;

(b) shear flow $\boldsymbol{u}_n = (u(x_2, x_3), 0, 0)$;

(c) circular flow $\boldsymbol{u}_n = c(x_2^2 + x_3^2)^{(n-1)/2}(0, -x_3, x_2)$ (n is an odd number).

Proofs of the propositions formulated below are very lengthy and are therefore not given in the present paper.

Proposition 1. *The higher-order term of degree* 2 *has the form of* (*a*), (*b*), *or*

$$\boldsymbol{u} = (\lambda x_1^2 + d(x_2^2 + x_3^2), -\lambda x_1 x_2, -\lambda x_1 x_3), \tag{2.3}$$

where λ *and* d *are arbitrary parameters.*

We note that, for constants λ and d, the function (2.3) is a steady-state axisymmetric solution of the Navier–Stokes equations with the pressure

$$p = \nu(2\lambda + 4d)x_1 + \lambda(-\lambda x_1^4/2 + d(x_2^2 + x_3^2)^2/4).$$

The streamlines are in meridional planes, and in the section $x_3 = 0$, the stream function is written as

$$\psi = \lambda x_1^2 x_2^2 + d x_2^4/2$$

(the volumetric flow rate $\psi = \text{const}$ in the stream tube is equal to $\pi\psi$). For $d = 0$, the slip condition $\boldsymbol{u} = 0$ is satisfied on the plane $x_1 = 0$, which can be treated as a local separation of the boundary layer. Figure 1 shows streamlines (curves 1–4) and the pressure dependence on x_1 (curve 5) for $\nu = 1/256$, $\lambda = 1$, and $d = 0$. For $\lambda d > 0$, the streamlines in the vicinity of the coordinate origin are similar to the streamlines in the case of flow over a body or a ring vortex (Fig. 2). However, in the case of flow over a body or a ring vortex, the streamlines from outside do not penetrate through the closed surface bounding the body or the ring vortex, whereas in the case considered, the external streamlines penetrate the entire interior.

Equation (2.2) is integrated once:

$$\boldsymbol{u} \times \boldsymbol{\omega} = \nabla P \qquad (\boldsymbol{\omega} = \nabla \times \boldsymbol{u}). \tag{2.4}$$

Here P is an arbitrary polynomial in $\boldsymbol{x}$.

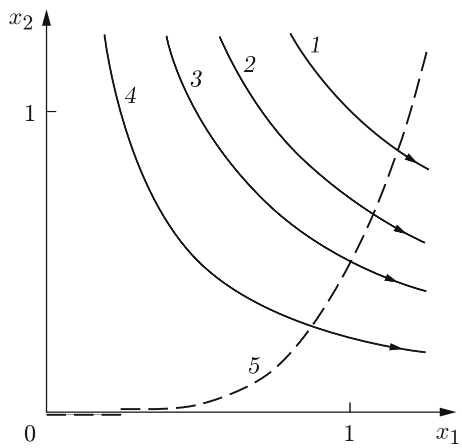

Fig. 1

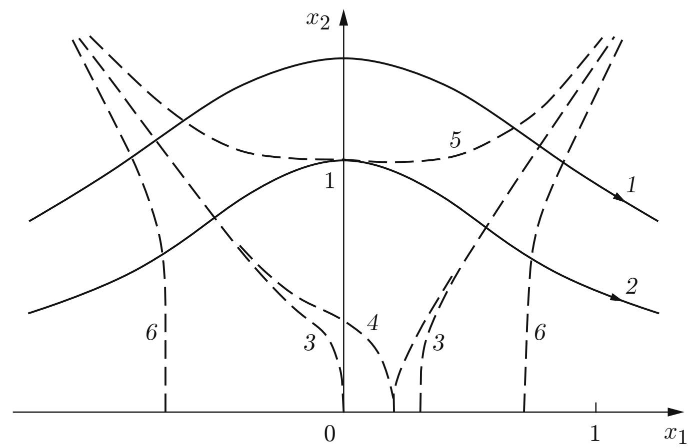

Fig. 2

Fig. 1. Streamlines (1–4) and dependence of the pressure on the coordinate x_1 (5) for $\nu = 1/256$, $\lambda = 1$, and $d = 0$: 1) $\psi = 1$; 2) $\psi = 1/2$; 3) $\psi = 1/4$; 4) $\psi = 1/16$; 5) $x_2 = -p(x_1)$.

Fig. 2. Streamlines (1 and 2) and pressure level lines (3–6) for $\nu = 1/256$, $\lambda = 1$, $d = 1/2$: 1) $\psi = 1$; 2) $\psi = 1/4$; 3) $p = 0$; 4) $p = 1/431$; 5) $p = 1/8$; 6) $p = -1/8$.

Let us consider the axisymmetric (with the symmetry axis x_1) higher-order term:

$$\boldsymbol{u} = (U_1, U_2x_2 - U_3x_3, U_2x_3 + U_3x_2).$$

Here U_i $(i = 1, 2, 3)$ are homogeneous polynomials in $x = x_1$ and $y = x_2^2 + x_3^2$ (the degree y is equal to two). Similarly, we determine the components Ω_i $(i = 1, 2, 3)$ of the axisymmetric vortex $\boldsymbol{\omega}$.

Proposition 2. *Nontrivial axisymmetric higher-order terms of degree $n \geq 3$ do not exist.*

3. Nonsolenoidal Solutions. The class of polynomial solutions of Eq. (2.4) is significantly extended if the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ is rejected. Nonsolenoidal solutions can be treated as steady-state flows of an ideal barotropic gas in a potential external-force field. Indeed, since in a barotropic gas, the pressure is a function of only density ρ, we have

$$(1/\rho)\nabla p = \nabla H(\rho),$$

where H is the enthalpy of the gas. Let a unit mass of the gas is acted upon by external force ∇V. Under these conditions, the equations of motion reduce to (2.4), where we should set

$$P = H(\rho) - V + (1/2)\boldsymbol{u} \cdot \boldsymbol{u}. \tag{3.1}$$

Given the solution $\boldsymbol{u}$, P of Eqs. (2.4), we find the density ρ from the continuity condition $\nabla \cdot (\rho \boldsymbol{u}) = 0$, and then, from equality (3.1), we determine the external-force potential V. Generally speaking, the functions ρ and V will not be polynomials since $\boldsymbol{u}$ is not the higher-order term of the local solution.

Proposition 3. *The nontrivial axisymmetric homogeneous polynomials of degree $n \geq 2$ that satisfy Eq.* (2.4) *are only the following:*

1) $U_1 = -(ax^2 + 2by)\Omega_3 - 2y(ax^2 + by)\partial\Omega_3/\partial y$, $U_2 = ax\Omega_3 + (ax^2 + by)\partial\Omega_3/\partial x$, $U_3 = 0$, *and* $P = (1/2)(ax^2 + by)y\Omega_3^2$, *where Ω_3 is a solution of the equation*

$$(1 - a - 4b)\Omega_3 = 3ax\,\frac{\partial\Omega_3}{\partial x} + (6ax^2 + 12by)\,\frac{\partial\Omega_3}{\partial y} + (ax^2 + by)\Big(\frac{\partial^2\Omega_3}{\partial x^2} + 4y\,\frac{\partial^2\Omega_3}{\partial y^2}\Big),$$

which exists if the arbitrary constants a and b satisfy some algebraic equation;

2) $U_1 = cx^{2m+1}$, $U_2 = 0$, $U_3 = dy^m$, $P = (m+1)(2m+1)^{-1}d^2y^{2m+1}$, *where c and d are arbitrary constants; $m = (n-1)/2 \geq 0$ is an integer.*

4. Solution of Degree 1. The solution of degree 1 is a linear function of $\boldsymbol{u} = B\boldsymbol{x}$, where B is a matrix. Substitution of this function into Eqs. (1.1) yields

$$\operatorname{sp} B \equiv \sum_i B_{ii} = 0, \qquad \dot{B} - \dot{B}^* + B^2 - B^{*2} = 0, \qquad p = -\frac{1}{2}\boldsymbol{x} \cdot (\dot{B} + B^2)\boldsymbol{x} - \frac{q}{2}, \tag{4.1}$$

where B^* is a transposed matrix and q is an arbitrary function of t. From this it follows that the symmetric part $a = (B + B^*)/2$ of the matrix B can be defined as an arbitrary function of time. Then, the antisymmetric part of $\Omega = (B - B^*)/2$ will be defined as a solution of the second equation in (4.1). The square of the traceless matrix of the second order is a symmetric matrix; therefore, Ω and, hence, the vortex in plane flow retain constant values. In the spatial case, the vortex can vary in magnitude and direction.

The solution of degree 1 satisfies the nonpenetration condition for a second-order surface arbitrarily deformed in time and is used in problems of the motion of a fluid of finite mass [4–12].

In the Kirchhoff elliptic vortex [13], the fluid velocity is continuous and equal to zero at infinity, the velocity vortex is constant inside the rotating ellipse S and equal to zero outside S. Because the solution of degree 1 satisfies the nonpenetration condition for the ellipse, it follows that $\boldsymbol{u} = B_0\boldsymbol{x}$ inside the ellipse S. In [14, 15], linear growth of the velocity at infinity is assumed. This condition is necessary for accounting for the inhomogeneity of the external flow. In the present paper, we also assume a tangential discontinuity of the velocity on S. In this generalized formulation, three-dimensional ellipsoidal vortices also exist. All these vortices were found in [16] and are of interest in connection with the Lavrent'ev turbulence model [17]. In the present paper, we calculate all plane elliptic vortices.

Thus, the fluid vortex $\omega = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$ is assumed to be a piecewise constant function of the point $\boldsymbol{x}$ which has a discontinuity on the ellipse S: $f(\boldsymbol{x}) \equiv \boldsymbol{x} \cdot A\boldsymbol{x} - 1 = 0$ (A is a symmetric positive definite matrix). For the plane flow considered, the value of the vortex does not depend on time. In the transformation to a coordinate system rotating at constant angular velocity, the velocity field outside S can be made potential. In this case, the equations of motion do not change if the pressure, which remains continuous, is defined appropriately. Thus, the fluid velocity is sought in the form

$$\boldsymbol{u} = \begin{cases} B_0\boldsymbol{x} & \text{inside } S, \\ a_1\boldsymbol{x} + \nabla\varphi & \text{outside } S, \end{cases} \qquad \nabla\varphi\Big|_\infty = 0,$$

where a_1 is a symmetric traceless matrix. We consider the nonpenetration condition for the ellipse

$$\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f = 0 \quad \text{in} \quad S.$$

According to the condition of nonpenetration from the interior, we obtain

$$\boldsymbol{x} \cdot \dot{A}\boldsymbol{x} + B_0\boldsymbol{x} \cdot 2A\boldsymbol{x} = \boldsymbol{x} \cdot (\dot{A} + AB_0 + B_0^*A)\boldsymbol{x} = 0 \qquad \text{at} \qquad f(\boldsymbol{x}) = 0$$

(the square matrix is symmetrized). Because the polynomial $f(\boldsymbol{x})$ has no multiple roots, the following equality holds for all $\boldsymbol{x}$:

$$\boldsymbol{x} \cdot (\dot{A} + AB_0 + B_0^*A)\boldsymbol{x} = cf(\boldsymbol{x}),$$

where c is a polynomial of degree 0, i.e., a number. Equating monomials of the same degree on the left and right sides of the equality, we obtain $c = 0$ and the matrix equation $\dot{A} + AB_0 + B_0^*A = 0$. Since the vector $A\boldsymbol{x}$ is orthogonal to S, the continuity condition for the normal velocity component is written as

$$\nabla\varphi \cdot A\boldsymbol{x} = B\boldsymbol{x} \cdot A\boldsymbol{x} \qquad \text{at} \qquad f(\boldsymbol{x}) = 0, \tag{4.2}$$

where $B = B_0 - a_1$. The pressure on S should be continuous. The pressure p_0 inside S is expressed in terms of the matrix B_0 by formula (4.1), and the pressure p_1 outside S is determined from the Cauchy–Lagrange integral.

Let us transform to the coordinate system (x_1', x_2') which is attached to the ellipse S. We denote the half-axes of the ellipse by $\sqrt{\alpha}$ and $\sqrt{\beta}$ and the angle between the half-axis $\sqrt{\alpha}$ (the x_1' axis) and the coordinate axis x_1 by ϑ. In this case, the coordinates of the point and the velocity are transformed by the formulas $\boldsymbol{x}' = U^*\boldsymbol{x}$ and $\boldsymbol{u}' = U^*\boldsymbol{u}$ and the matrix functions are transformed as $B_0' = U^*B_0U$, from which it follows that $U^*\dot{B}_0U = \dot{B}_0' + PB_0' - B_0'P$, where

$$U = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}, \qquad P = U^*\dot{U} = \begin{pmatrix} 0 & -\dot{\vartheta} \\ \dot{\vartheta} & 0 \end{pmatrix}.$$

The velocity circulation $2\pi\gamma$ outside the ellipse S and the values of the vortex $\omega = \omega_0$ inside S and $\omega = 0$ outside S do not change. Multiplication of the equation for the matrix A on the left by U^* and on the right by U gives

$$\dot{A} + A(B_0 - P) + (B_0^* + P)A = 0 \tag{4.3}$$

(primes are omitted). The pressure continuity condition on S becomes

$$\frac{\partial\varphi}{\partial t} + \boldsymbol{x}\cdot(a_1 + P)\nabla\varphi + \frac{1}{2}\,|\nabla\varphi|^2 + \frac{1}{2}\,\boldsymbol{x}\cdot(\dot{a}_1 + 2Pa_1 + a_1^2$$
$$- \dot{B}_0 - PB_0 + B_0P - B_0^2)\boldsymbol{x} - \frac{q}{2} = 0 \quad \text{at} \quad f(\boldsymbol{x}) = 0. \tag{4.4}$$

The boundary condition (4.2) remains unchanged.

Problem (4.2)–(4.4) is further solved in the indicted moving coordinate system in which the matrix A is diagonal:

$$A = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \beta^{-1} \end{pmatrix}.$$

For the given A and B, the harmonic function φ is defined by conditions (4.2), $\nabla\varphi\Big|_\infty = 0$, and the value of the circulation $2\pi\gamma$ to within an insignificant constant term. All these conditions are satisfied by a function of the form

$$\varphi = \frac{1}{2}\Big(x_1^2 - x_2^2 - \frac{\alpha-\beta}{2}\Big)v_1(\lambda) + x_1x_2v_2(\lambda) + \frac{1}{2}\,v_3(\mu) = \frac{1}{2}\,\Big(\boldsymbol{x}\cdot V(\lambda)\boldsymbol{x} - v_0(\lambda) + \frac{1}{2}\,v_3(\mu)\Big),$$

where

$$V(\lambda) = \begin{pmatrix} v_1 & v_2 \\ v_2 & -v_1 \end{pmatrix}, \qquad v_0(\lambda) = \frac{\alpha-\beta}{2}\,v_1(\lambda).$$

Here $\lambda \le \mu$ are elliptic coordinates which are defined as solutions of the following algebraic equation for λ:

$$\frac{x_1^2}{\alpha-\lambda} + \frac{x_2^2}{\beta-\lambda} - 1 = 0. \tag{4.5}$$

On S, the coordinate $\lambda = 0$, and outside S $\lambda < 0$; the values of μ lie between the values α and β. The functions v_k satisfy the ordinary differential equations which are derived by differentiation of equality (4.5) and are easily integrated:

$$v_1(\lambda) = v_1'(0)d_1^2(\alpha\beta)^{1/2}(d_1-\lambda)^{-1}(d_1 - \lambda + \theta(\lambda)^{1/2})^{-1},$$
$$v_2(\lambda) = v_2'(0)(\alpha\beta)^{3/2}\theta(\lambda)^{-1/2}(d_1 - \lambda + \theta(\lambda)^{1/2})^{-1},$$
$$v_3(\mu) = \gamma\arcsin((2\mu-\alpha-\beta)/(\alpha-\beta)), \qquad v_3'(\mu) = \gamma(-\theta(\mu))^{-1/2}.$$

Here $d_1 = (\alpha+\beta)/2$ and $\theta(\lambda) = (\alpha-\lambda)(\beta-\lambda)$. The variation in μ from β to α corresponds to the circulation around quarter of the contour enclosing S; in this case, the term $v_3(\mu)/2$ gains an increment $\pi\gamma/2$, and, in circulation around the entire contour, $2\pi\gamma$ undergoes a discontinuity and the remaining terms are continuous.

A point of the ellipse S is a function of the elliptic coordinate μ ($\lambda = 0$). The equation of S and the definition of μ imply the equalities

$$x_1^2 = (\alpha-\beta)^{-1}\alpha(\alpha-\mu), \qquad x_2^2 = (\alpha-\beta)^{-1}\beta(\mu-\beta) \qquad \text{in} \quad S. \tag{4.5a}$$

Boundary condition (4.2) can be reduced to a matrix equation similar to (4.3). Differentiating definition (4.5) and taking into account equality (4.5a), we obtain the formula

$$\nabla\varphi\Big|_S = V(0)\boldsymbol{x} - 2\varphi_0'\,\frac{A\boldsymbol{x}}{|A\boldsymbol{x}|^2} + \gamma(\alpha\beta)^{-1/2}\,\frac{JA\boldsymbol{x}}{|A\boldsymbol{x}|^2}, \tag{4.6}$$

where

$$\varphi_0' = \frac{1}{2}\,(\boldsymbol{x}\cdot V'(0)\boldsymbol{x} - v_0'(0)), \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since the lines $\mu = \text{const}$ are orthogonal to S, the term v_3 is eliminated by substitution of φ into (4.2):

$$A(V(0) - B) + (V(0) - B^*)A - 2(V'(0) - v_0'(0)A) = 0. \tag{4.7}$$

Writing this symmetric matrix equation elementwise, we obtain three scalar equations, two of which are independent and uniquely define the unknown quantities $v_1'(0)$ and $v_2'(0)$.

Let us write the elements of the matrix equations (4.3) and (4.7) by taking into account the equality $B = B_0 - a_1$:

$$\dot{\alpha} = 2a_{0,11}\alpha, \qquad \dot{\beta} = -2a_{0,11}\beta,$$

$$(a_{0,12} - \omega_1 + \dot{\vartheta})/\alpha + (a_{0,12} + \omega_1 - \dot{\vartheta})/\beta = 0, \quad a_{11} - v_1(0) + (\alpha + \beta)v_1'(0)/2 = 0, \tag{4.8}$$

$$(v_2(0) - a_{12} + \omega_1)/\alpha + (v_2(0) - a_{12} - \omega_1)/\beta - 2v_2'(0) = 0.$$

Here $B_0 = a_0 + \Omega_0$, $a = a_0 - a_1$, $\omega_1 = \omega_0/2$, and

$$\Omega_0 = \begin{pmatrix} 0 & -\omega_1 \\ \omega_1 & 0 \end{pmatrix}.$$

Differentiation of (4.5) in view of (4.5a) yields

$$\frac{\partial \varphi}{\partial t}\Big|_S = \frac{1}{2}\left(\boldsymbol{x} \cdot \dot{V}(0)\boldsymbol{x} - \dot{v}_0(0)\right) - \varphi_0' \frac{\boldsymbol{x} \cdot \dot{A}\boldsymbol{x}}{|A\boldsymbol{x}|^2} + \frac{\gamma}{2}(\dot{\alpha} - \dot{\beta})(\alpha\beta)^{-3/2} \frac{x_1 x_2}{|A\boldsymbol{x}|^2}. \tag{4.9}$$

Substituting expressions (4.6) and (4.9) into boundary condition (4.4) and eliminating $\dot{\alpha}$, $\dot{\beta}$, a_0, and a_1 by means of (4.8), we obtain

$$G_1 + (x_1 x_2 K_1 + K_2)/|A\boldsymbol{x}|^2 = 0 \qquad \text{at} \qquad f(\boldsymbol{x}) = 0, \tag{4.10}$$

where G_1, K_1, and K_2 are homogeneous polynomials in $\boldsymbol{x}$ of degrees 2, 2, and 4, respectively (the expressions for them are lengthy and not given here); K_1 and K_2 depend only on x_1^2 and x_2^2. The left side of condition (4.10) is a homogeneous function; therefore, the equality

$$x_1 x_2 K_1 + K_2 = |A\boldsymbol{x}|^2(a x_1 x_2 + b x_1^2 + c x_2^2) \qquad \text{at} \qquad x_1 x_2 K_1' + K_2' = 0$$

is valid everywhere. Because K_1' and K_2' are polynomials of x_1^2 and x_2^2 and the function $x_1 x_2$ depends irrationally on x_1^2 and x_2^2, it follows that $K_1' = 0$, $K_2' = 0$, i.e., the polynomials K_1 and K_2 are divided by $|A\boldsymbol{x}|^2$. This implies that

$$v_1'(0) = 0, \qquad v_2'(0) = -\gamma(\alpha - \beta)(\alpha\beta)^{-3/2}/2. \tag{4.11}$$

Then, $v_1 = 0$ and $K_1 = 0$, and the last two equations of (4.8) lead to

$$a_{11} = 0, \qquad a_{12} = \frac{\alpha - \beta}{\alpha + \beta}\left(2\gamma(\sqrt{\alpha} + \sqrt{\beta})^{-2} - \omega_1\right). \tag{4.11a}$$

In view of formulas (4.6) and (4.9), boundary condition (4.4) reduces to the matrix equation

$$\dot{V}(0) + (a_1 + P)V(0) + V(0)(a_1 - P) + V(0)^2 - \frac{\gamma^2}{\alpha\beta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$- \dot{a} - 2Pa - aa_0 - a_0 a + a^2 - \Omega_0^2 - qA = 0. \tag{4.12}$$

Because the matrices $V(0)$ and a have zero diagonal elements according to (4.11) and (4.11a), we have $\dot{V}(0) - \dot{a} = 0$ due to the diagonality of the sums of the remaining terms of Eqs. (4.12). From this, it follows that

$$v_2(0) - a_{12} = \frac{\alpha - \beta}{\alpha + \beta}\Big(\omega_1 - \frac{\gamma}{\sqrt{\alpha\beta}}\Big) = \text{const}.$$

Multiplying Eq. (4.12) on the right by A^{-1} and taking the difference of the diagonal elements, we eliminate the arbitrary parameter q. In view of (4.8), the result becomes

$$\frac{\alpha - \beta}{\alpha + \beta}\Big(\omega_1 - \frac{\gamma}{\sqrt{\alpha\beta}}\Big)\Big(- 2\dot{\vartheta} + \omega_1 + \frac{\gamma}{\sqrt{\alpha\beta}}\Big) = 0.$$

Thus, ignoring the trivial case $\alpha \equiv \beta$, the following statement is proved.

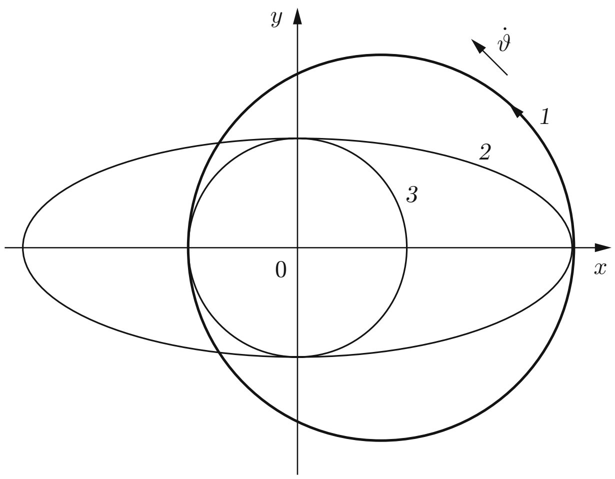

Fig. 3

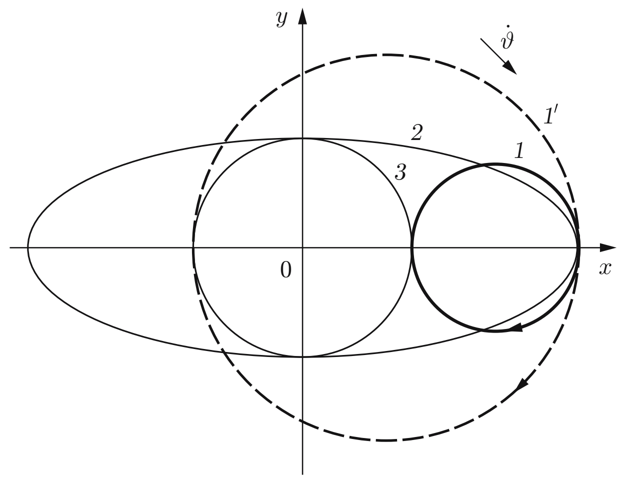

Fig. 4

Fig. 3. Kirchhoff flow ($a_1 = 0$; the velocity field is continuously): 1) fluid particle trajectory; 2) ellipse S at the initial time; 3) circle inscribed in the ellipse.

Fig. 4. Discontinuous flow ($a_1 = 0$): 1) trajectories of an internal fluid particle; 1′) trajectories of an external fluid particle; 2) ellipse S at the initial time; 3) circle inscribed in the ellipse.

Proposition 4. *Let the vortex ω be equal to $\omega_0 = 2\omega_1$ inside the ellipse S: $x_1^2/\alpha + x_2^2/\beta = 1$ and be equal to zero outside S, and let the fluid velocity $\boldsymbol{u} \to a_1\boldsymbol{x}$ as $|\boldsymbol{x}| \to \infty$ ($a_1 = a_1^*$ and $\operatorname{sp} a_1 = 0$). Then, inside S, we have $\boldsymbol{u} = B_0\boldsymbol{x}$, where*

$$B_0 = \begin{pmatrix} a_{0,11} & a_{0,12} - \omega_1 \\ a_{0,12} + \omega_1 & -a_{0,11} \end{pmatrix},$$

and only the following solutions exist [in the moving coordinate system (x_1, x_2)]:

1) *a matrix a_1 an arbitrary function of time t, $\dot\alpha = 2a_{0,11}\alpha$, $\alpha\beta = \text{const}$, $\dot\vartheta = \omega_1 + (\alpha+\beta)(\alpha-\beta)^{-1}a_{0,12}$, $a_{0,11} = a_{1,11}$, $a_{0,12} = a_{1,12} + a_{12}$, and $2\pi\gamma = \omega_0\pi\sqrt{\alpha\beta}$ velocity circulation around S;*

2) *α, β, $\dot\vartheta$, a_0, and a_1 are constants, $\dot\vartheta = \omega_0/4 + \gamma/(2\sqrt{\alpha\beta})$, $a_{0,11} = a_{1,11} = 0$, $a_{0,12} = (\alpha-\beta) \times (\alpha+\beta)^{-1}(-\omega_0/4 + \gamma/(2\sqrt{\alpha\beta}))$, $a_{1,12} = a_{0,12} - a_{12}$, where a_{12} is given by expression* (4.11a).

In the first solution, the fluid velocity is continuous on S. For $a_1 = 0$, this solution becomes the Kirchhoff solution [13], and for $a_1 \neq 0$, it includes, as particular cases, the solutions obtained in [14, 15]. In the second solution, the fluid velocity undergoes a tangential discontinuity on S.

Figure 3 shows the fluid particle trajectory corresponding to the Kirchhoff solution. Curve 1 shows the fluid particle trajectory relative to the motionless coordinate system with origin $(\sqrt{\alpha}, 0)$ during the half-turn of the ellipse S. Curve 2 shows the ellipse at the initial time, and curve 3 the circle inscribed in it. The direction of the angular velocity of rotation of the ellipse $\dot\vartheta$ for $\omega_0 > 0$ is specified.

Figure 4 shows the trajectories of the external and internal fluid particles corresponding to the discontinuous solution (with $a_1 = 0$). Curve 1 shows the trajectory of an internal fluid particle, and curve 1′ the trajectory of the external particle having the same initial position during the half-turn of the ellipse S. The direction of rotation of the ellipse is indicated for $\omega_0 > 0$. For both flows, the fluid particle trajectories have the shape of a circle.

As an example of the first solution, we consider the case where the matrix a_1 is constant in the motionless coordinate system (x, y), is diagonal, and contains elements b and $-b$ on the diagonal. According to (4.1), the pressure is equal to $p = -b^2(x^2+y^2)/2 - q/2$, i.e., an anticyclone occurs. Flow with this matrix arises at the site of collision of two flows. In this flow, an elliptic vortex described by the first solution can exist. In the moving coordinate system, we have

$$a_1 = U^* \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} U = b \begin{pmatrix} \cos(2\vartheta) & -\sin(2\vartheta) \\ -\sin(2\vartheta) & -\cos(2\vartheta) \end{pmatrix}.$$

In this case, the system of equations is solved in quadratures. For the variable $\delta = (\alpha - \beta)(\alpha\beta)^{-1/2}/2$, we obtain

$$\dot{\delta} = 2b\sqrt{1+\delta^2}\cos(2\vartheta), \qquad b\omega_1^{-1}\sin(2\vartheta) = \delta^{-1}(c + \ln(1+\sqrt{1+\delta^2})) \equiv f_0(\delta, c),$$

where the constant c is determined from the initial conditions $\delta = \delta_0$ and $\vartheta = \vartheta_0$ at $t = 0$.

The set $M = \{\delta\colon\ |f_0(\delta, c)| < b\omega_1^{-1}\}$ consists of nonintersecting open intervals. The isolated points of the complement $\mathbb{R} \setminus M$ are points of equilibrium. If the initial point δ_0 falls in a finite interval of the set M the ellipse S rotates and pulsates, remaining bounded. If δ_0 belongs to an infinite interval, the motion is aperiodic: the ellipse is extended infinitely in the direction of the outgoing jet or tends to the position of equilibrium.

5. Solution of Degree 3 with a Shear Higher-Order Term. Let us find the solution of degree $n = 3$ with a shear higher-order term of the particular form

$$\boldsymbol{u}_3 = \begin{pmatrix} ay^3 \\ 0 \\ 0 \end{pmatrix}, \qquad a \neq 0. \tag{5.1}$$

The coordinates of a point in space will be denoted by x, y, and z.

Proposition 5. *The solution of degree 3 polynomial in the coordinates with the higher-order term* (5.1) *has the form*

$$\boldsymbol{u} = \begin{pmatrix} -(b_1 + c_1')x + a_1 y + a_1' z + a_2 y^2 + a_2' yz + a_2'' z^2 + ay^3 \\ b_1 y + b_1' z \\ -2o_2 x + c_1 y + c_1' z + c_2 y^2 \end{pmatrix}, \tag{5.2}$$

where the coefficients depend on t and satisfy the following equations:

$$3ab_1' + 2c_2 a_2'' = 0, \quad (o_3 - c_2^2/(3a))a_2'' = 0, \quad o_2 a_2'' = 0, \quad o_2 a_2' = 0,$$

$$\begin{aligned}
&\dot{a} + a(2b_1 - c_1') + a_2' c_2 = 0,\\
&\dot{a}_2 + c_2(-2o_2 + a_1') + a_2(b_1 - c_1') + a_2' c_1 = 0,\\
&\dot{a}_2' + 2a_2 b_1' + 2a_2'' c_1 = 0,\\
&\dot{a}_2'' + a_2' b_1' + a_2''(-b_1 + c_1') = 0,\\
&\dot{c}_2 + o_3 a_2' + c_2(2b_1 + c_1') = 0,\\
&\dot{a}_1 + 2\dot{o}_3 + 2(2o_1 + b_1' - c_1)o_2 - (2o_3 + a_1)c_1' + a_1' c_1 - 6a\nu = 0,\\
&\dot{a}_1' + (2o_3 + a_1)b_1' - a_1' b_1 = 0,\\
&\dot{c}_1 - \dot{b}_1' - 2\dot{o}_1 + 2a_1' o_3 + (b_1 + c_1')(-2o_1 - b_1' + c_1) = 0.
\end{aligned} \tag{5.3}$$

Remark 1. For $\boldsymbol{o} = 0$, solution (5.2) is included in the class of "nearly plane" flows

$$\boldsymbol{u} = \Big(-x\Big(\frac{\partial v(y,z,t)}{\partial y} + \frac{\partial w(y,z,t)}{\partial z}\Big) + u'(y,z,t),\ v(y,z,t),\ w(y,z,t)\Big),$$

which also contains higher-degree solutions polynomial in $\boldsymbol{x}$.

System (5.3) breaks up into two systems.

System I. $o_2 = 0$; $o_3 = c_2^2/(3a)$. The quantities o_1, b_1, and c_1' are arbitrary functions of t. The system has the eighth order.

System II. $a_2' = a_2'' = b_1' = 0$. The angular velocity of rotation of the coordinate system $\boldsymbol{o}$ and the coefficients b_1 and c_1' are arbitrary functions of t. The system has the sixth order.

Given the arbitrary functions of t, the solution of the nonlinear system reduces to the consecutive solution of the linear subsystems. Naturally, in physical problems, it is necessary to specify the coefficients at terms of higher-order degrees rather than the indicated arbitrary elements. Then, the arbitrary quantities can be determined.

An example of solution breakdown is given below. In system II, let the arbitrary elements $\boldsymbol{o} = 0$ and $c_1' = 0$ be specified. We consider the particular solution of this system ($a_2 = c_2 = 0$):

$$\dot{a} + 2ab_1 = 0,$$
$$\dot{a}_1 + a_1' c_1 - 6a\nu = 0,$$
$$\dot{a}_1' - a_1' b_1 = 0, \tag{5.4}$$
$$\dot{c}_1 + b_1 c_1 = 0.$$

System (5.4) is a simple system containing viscosity ν. For an arbitrary given function $b_1(t)$, the solution of system (5.4) generates a solution of the Navier–Stokes equations. Setting $b_1(t) \to \infty$ as $t \to t_0$, we obtain a solution which breaks down at the final time t_0. We can also specify b_1 as a function of a, a_1, a_1', and c_1. Then, system (5.4) is closed. For $b_1 = -a$, or $b_1 = a_1'$, or $b_1 = -c_1$, the solution also breaks down. For definiteness, let $b_1 = a_1'$. Then,

$$a = A(t - t_0)^2, \qquad a_1 = 2\nu A(t - t_0)^3 + Ct + B,$$

$$a_1' = 1/(t_0 - t), \qquad c_1 = C(t - t_0),$$

where t_0, A, B, and C are arbitrary constants. It is necessary to note that the viscosity and its sign do not influence the breakdown of the solution.

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